Variational Principle of Melting Elastic Solid with Thermal Layer.

Yoshiyuki FURUYA

Department of Applied Mathematics, Faculty of Engineering, Toyama University, Takaoka

A mathematical method is proposed for investigating the melting elastic solid with thermal layer; the variational method together with the enlarged heat balance integral provide the series solution of the position of melting line and the thickness of thermal layer. The technique is powerful because it does not require the evaluation of temperature field. The quadratic equation of temperature field has used as a test function.

§1. Introduction

In the previous papers,[1),2)] the author analysed the melting elastic solid. A melting slab occupying the region of x axis exposed to a prescrided heat input. One case the melted portion was immedeately removed[2)] and the other case the melted portion was not removed. In both investigations, the variational method of heat conduction and thermal stress was used.[3)~6)]

In this paper, we shall concerned with the problem that the melting begins before the temperature change reached the erd $x=L$. The melted portion is not removed. This case the thermal layer is formed. Therefore, the thermal field has two parameters, i. e. the position of melting line and the thickness of thermal layer. So that, we handle this problem by the next procedure:

1. The variational invariants of this problem are introduced and the variational equation is formulated.
2. An extension of heat balance integral[7),8)] is introduced.
3. The variational equation together with the extension of heat balance integral leads the series solution of the position of melting line $s(t)$,and the thickness of thermal layer $q(t)$.

§2. Basic Equations.

The problem of melting slab be considered. Consider a slab occupying the region of x axis, exposed to a prescribed heat input $Q(t)$ at $x=0$. The melted portion is not removed. It will be assumed here that the face $x=0$ is suddenly exposed, at $t=0$, to the heat input Q_0; the heat input is constant before melting begins. The thickness of the medium which has melted is denoted by $s(t)$. Let $T+\theta$ denote the temperature of unit element, θ being the temperature change.

A region in the solid state $s(t)<x<s(t)+q(t)$ is defined to de thermal layer; for

$x>s(t)+q(t)$, the slad is at equilibrium temperature and there is no heat transfer beyond this point.

We shall denote the heat conduction equation and the law of thermal expansion as follows:

$$-\lambda\frac{\partial\theta}{\partial x}=\dot{H}, \qquad c\theta=-\frac{\partial H}{\partial x}, \tag{2.1}$$

H······quantity which represents the rate of heat flow by $\dot{H}$,
c······heat capacity per unit volume,

$$\alpha\theta=\frac{\partial\xi}{\partial x}, \tag{2.2}$$

α······coefficient of thermal expansion,
ξ······displacement.

The boundary condition at $x=s(t)$ is expressed as[9]

$$H_S=H_M-\rho ls, \quad at \quad x=s(t). \tag{2.3}$$

ρl······latent heat per unit volume.

By taking the origin of the time as the time when melting begins, i. e. $s(0)=0$, the formula (2.3) becomes

$$H_S=H_M-\rho ls, \quad at \quad x=s(t). \tag{2.4}$$

By refering the Biot's paper;[3)~5)] near thermal equilibrium state, the temperature is expressed by a quadratic form of the coordinate, we take the temperature of melted state:

$$\theta_M=\theta_m+A(t)(s-x)^2. \tag{2.5}$$

suffix M······melted state.
suffix m······melting state.

Substituting eq. (2.5) into eq. (2.1_2) and integrating, we have

$$H_M=-c_M\left[\theta_m x-\frac{A(t)}{3}(s-x)^3\right] \tag{2.6}$$

The boundary condition at $x=0$ is

$$H(0,t)=\int_0^t Q(t)at+\Delta. \tag{2.7}$$

Δ······heat transported to the right across a unit cross sectional area at $x=0$, before melting begins.

Therefore, $A(t)$ in eq. (2.6) is determined as

$$A(t)=\frac{3Q_I}{c_m s^3}, \tag{2.8}$$

with

$$Q_l = \int_o^t Q(t)dt + \Delta. \tag{2.9}$$

From eqs. (2.5) and (2.8), we have

$$\theta_M = \theta_m + \frac{3\theta_l}{c_m s^3}(s-x)^2. \tag{2.10}$$

For the solid state, we take the temperature profile as

$$\theta_S = a(t) + b(t)(x-s) + c(t)(x-s)^2.$$

suffix S ······solid state.

The conditions

$$\theta_S(s, t) = \theta_m , \quad \theta_S(s+q, t) = 0. \quad \frac{\partial\theta_S(s+q, t)}{\partial x} = 0. \tag{2.11}$$

provide $a(t)$, $b(t)$ an $c(t)$:

$$\theta_S = \theta_m \left[1 - \frac{2}{q}(x-s) + \frac{1}{q^2}(x-s)^2 \right] . \tag{2.12}$$

§3. Variational Principle.

Refering to the previous paper, we introduce the variational invariants

$$V = \int_o^s \frac{1}{2}\left(\frac{c_M}{T} + E_M\alpha_M^2\right)\theta^2\, dx + \int_s^{s+q} \frac{1}{2}\left(\frac{c_S}{T} + E_S\alpha_S^2\right)\theta_S^2\, dx. \tag{3.1}$$

$$D = \mathrm{p}\left[\int_o^s \frac{1}{2\lambda_M T} H_M^2\, dx + \int_s^{s+q} \frac{1}{2\lambda_S T} H_S^2\, dx\right] . \tag{3.2}$$

We take the variations as the changes of the quantities due to the virtual displacement of the melting line $s(t)$; we assume θ, c, E, α, λ are continuous.

By use of eqs. (2.1), (2.2) and (2.4), we may evaluate the variations of V and D as follows:

$$\begin{aligned}
\delta V = & \int_o^s \left(\frac{1}{T}c_M\theta_M\,\delta\theta_M + E_M\alpha_M^2\,\theta_M\,\delta\theta_M\right)dx + \int_s^{s+q}\left(\frac{1}{T}c_S\theta_S\,\delta\theta_S + E_S\alpha_S^2\,\theta_S\,\delta\theta_S\right)dx = \\
& -\int_o^s \frac{1}{T}\theta_M\,\frac{\partial}{\partial x}(\delta H_M)dx + \int_o^s E_M\,\alpha_M\,\theta_M\,\frac{\partial}{\partial x}(\delta\xi_M)dx - \int_s^{s+q}\frac{1}{T}\theta_S\,\frac{\partial}{\partial x}(\delta H_S)dx \\
& + \int_s^{s+q} E_S\alpha_S\theta_S\,\frac{\partial}{\partial x}(\delta\xi_S)dx = -\frac{1}{T}\Big[\theta_M\delta H_M\Big]_o^s + \int_o^s \frac{1}{T}\frac{\partial\theta_M}{\partial x}\delta H_M\,dx - \frac{1}{T}\Big[\theta_S\delta H_S\Big]_s^{s+q} \\
& + \int_s^{s+q}\frac{1}{T}\frac{\partial\theta_S}{\partial x}\,\delta H_S\,dx + \Big[E_M\alpha_M\,\theta_M\,\delta\xi_M\Big]_o^s - \int_o^s \frac{\partial}{\partial x}(E_M\alpha_M\,\theta_M)\delta\xi_M\,dx \\
& + \Big[E_S\,\alpha_S\,\theta_S\Big]_s^{s+q} - \int_s^{s+q}\frac{\partial}{\partial x}(E_S\alpha_S\,\theta_S)\delta\xi_S\,dx
\end{aligned}$$

$$\therefore \delta V = -\frac{\rho l \theta_m}{T}\delta s - (E_M \alpha_M \theta_M \delta\xi_M)_{x=0} - \int_o^s \frac{1}{\lambda_M T}\dot{H}_M \,\delta H_M \,dt - \int_s^{s+q} \frac{1}{\lambda_S T}\dot{H}_S \,\delta H_S \,dx$$

$$-\int_o^s \frac{\partial}{\partial x}(E_M \alpha_M \theta_M)\delta\xi_M \,dx - \int_s^{s+q}\frac{\partial}{\partial x}(E_S \alpha_S \theta_S)\delta\xi_S \,dx, \tag{3.3}$$

$$\delta D = \mathrm{p}\left\{\frac{1}{2\lambda_m T}\left[H_m(\dot{s}, t)^2 - H_m(s, t)^2\right] + \int_o^s \frac{1}{\lambda_M T} H_M \delta H_M \,dx\right.$$

$$\left. + \int_s^{s+q}\frac{1}{\lambda_S T} H_S \delta H_S \,dx\right\}$$

$$= \mathrm{p}\left[\frac{1}{2\lambda_m T}\rho l s(2H_M(s, t) - \rho l s)\right] + \int_o^s \frac{1}{\lambda_M T}\dot{H}_M \delta H_M \,dx$$

$$+ \int_s^{s+q}\frac{1}{\lambda_S T}\dot{H}_S \,\delta H_S \,dx,$$

$$\therefore \delta D = \frac{1}{\lambda_m T}\left[\rho l s \; H_M(s, t) - \rho^2 l^2 s\dot{s} + \rho l s\left(\frac{\partial H_M(s, t)}{\partial x}\dot{s} + \frac{\partial H_M(s, t)}{\partial t}\right)\right]$$

$$+ \int_o^s \frac{1}{\lambda_M T}\dot{H}_M \delta H_M \,dx + \int_s^{s+q}\frac{1}{\lambda_S T}\dot{H}_S \,\delta H_S \,dx. \tag{3.4}$$

The equations (3.3) and (3.4) provide the variational equation

$$\delta V + \delta D = -\frac{\rho l \theta_m}{T}\delta s - (E_M \alpha_M \theta_M \delta\xi_M)_{x=0} - \int_o^s \frac{\partial}{\partial x}(E_M \alpha_M \theta_M)\delta\xi_M \,dx$$

$$-\int_s^{s+q}\frac{\partial}{\partial x}(E_S \alpha_S \theta_S)\delta\xi_S \,dx + \frac{1}{\lambda_m T}\left[\rho l\dot{s} H_M(s. t) - \rho^2 l^2 s\dot{s}\right.$$

$$\left. + \rho l s\left(\frac{\partial H_M(s. t)}{\partial x}\dot{s} + \frac{\partial H_M(s. t)}{\partial t}\right)\right]\delta s. \tag{3.5}$$

We shall evaluate the variational equation (3.5) by inserting the quadratic equations (2.10) and (2.12); for the sake of simplicity we take c, E, α, λ etc. as constants.

From eqs. (2.10) and (2.12), we have:

$$\int_o^s \theta_M^2 \,dx = \theta_m^2 \, s + \frac{2\theta_m Q_I}{c_M} + \frac{9Q_I^2}{5c_M^2}\,\frac{1}{s}, \qquad \int_s^{s+q}\theta_S^2 \,dx = \frac{1}{5}\theta_m^2 \, q,$$

$$\therefore \; V = \frac{1}{2}\left(\frac{c_M}{T} + E_M \alpha_M^2\right)\left(\theta_m^2 \, s + \frac{2\theta_m Q_I}{c_M} + \frac{9Q_I^2}{5c_M^2}\,\frac{1}{s}\right) + \frac{1}{10}\left(\frac{c_S}{T} + E_S \alpha_S^2\right)\theta_m^2 \, q, \tag{3.6}$$

$$\therefore \delta V = \frac{1}{2}\left(\frac{c_M}{T} + E_M \alpha_M^2\right)\left(\theta_m^2 - \frac{9\theta^2}{5c_M^2}\,\frac{1}{s^2}\right)\delta s + \frac{1}{10}\left(\frac{c_S}{T} + E_S \alpha_S^2\right)\theta_m^2 \,\delta q, \tag{3.7}$$

The equations (2.8) and (2.6) provide

$$H_M = -c_M\left[\theta_m \, x - \frac{Q_I}{c_M \, s^3}(s - x)^3\right] \quad . \tag{3.8}$$

The equations (2.1_2) and (2.12) and the condition that is derived from the definition of thermal layer

$$H(s+q,\ t)=0 \tag{3.9}$$

provide

$$H_S=-c_S\theta_m\left[(x-s)-\frac{1}{q}(x-s)^2+\frac{1}{3q^2}(x-s)^3-\frac{1}{3}q\right]. \tag{3.10}$$

From eqs. (3.8) and (3.10), we may evaluate D as follows:

$$\int_0^s H_M^2\,dx=c_M^2\left(\frac{\theta^2{}_m}{3}s^3-\frac{\theta_m Q_I}{10c}s^2+\frac{Q_I^2}{7c_M^2}s\right).$$

$$\int_s^{s+q} H_S^2\,dx=\frac{c^2{}_S\,\theta^2{}_m}{63}q^3.$$

$$\therefore\ D=\mathrm{p}\left[\frac{c_M^2}{2\lambda_M T}\left(\frac{\theta^2{}_m}{3}s^3-\frac{\theta_m Q_I}{10c_M}s^2+\frac{Q_I^2}{7c_M^2}s\right)+\frac{c^2{}_S\,\theta^2{}_m}{2\lambda_S\,T\times 63}q^3\right] \tag{3.11}$$

$$\therefore\delta D=\mathrm{p}\left[\frac{c^2{}_M}{2\lambda_M T}\left(\theta^2{}_m s^2-\frac{\theta_m Q_I}{5c_M}s+\frac{Q^2{}_I}{7c^2{}_M}\right)\delta s+\frac{c^2{}_S\,\theta^2{}_m}{2\lambda_S T\times 21}q^2\,\delta q\right].$$

$$\therefore\delta D=\frac{c^2{}_M}{2\lambda_M T}\left[2\theta^2{}_m s\dot{s}-\frac{\theta_m}{5c_M}(Q(t)s+Q_I\,s)+\frac{2Q_I\,Q(t)}{7c^2{}_M}\right]\delta s$$

$$+\frac{c^2{}_S\,\theta^2{}_m}{21\lambda_S\,T}q\,\dot{q}\ \delta q. \tag{3.12}$$

From eqs. (2.2) and (2,10), we see:

$$\xi_M=\alpha_M\left[\theta_m\,x-\frac{Q_I}{c_M s^3}(s-x)^3\right]. \tag{3.13}$$

$$\therefore\delta\xi_M=-\frac{3Q_I\,\alpha_M}{c_M}\ \frac{x(s-x)^2}{s^4}\delta s, \tag{3.14}$$

$$\therefore(\delta\xi_M)_{x=0}=0. \tag{3.15}$$

From eqs. (2.2) and (2.12), we see:

$$\xi_S=\alpha\theta_m\left[(x-s)-\frac{1}{q}(x-s)^2+\frac{1}{3q^2}(x-s)^3-\frac{1}{3}q\right], \tag{3.16}$$

$$\therefore\delta\xi_S=-\alpha\theta_m\left[1-\frac{2}{q}(x-s)+\frac{1}{q^2}(x-s)^2\right]\delta s. \tag{3.17}$$

Using eqs. (2.10), (3.14), (2.12), (3.17), we find:

$$\int_0^s\frac{\partial\theta_M}{\partial x}\delta\xi_M\,dx=\frac{9Q^2{}_I\,\alpha_M}{10c^2{}_M\,s^2}\ \delta s,\qquad\int_s^{s+q}\frac{\partial\theta_S}{\partial x}\ \delta\xi\ \ dx=\frac{1}{2}\alpha_S\,\theta^2_m\ \delta s. \tag{3.18}$$

The equation (3.8) provides:

$$H_M(s.t)=-c_M\ \theta_m\ s,\qquad \frac{\partial H_M(s.t)}{\partial x}=-c_M\ \theta_m,\quad \frac{\partial H_M(s.t)}{\partial t}=0\ . \qquad (3.19)$$

Substituting eqs. (3.7), (3.12), (3.15), (3.18) and (3.19) into the variational equation (3.5), we find

$$\left\{\frac{1}{2}\left(\frac{c_M}{T}+E_M\alpha_M^2\right)\left(\theta_m^2-\frac{9Q_l^2}{5c_M^2}\ \frac{1}{s^2}\right)+\frac{c_M^2}{2\lambda_M T}\left[2\theta_m^2 s\dot{s}-\frac{\theta_m}{5c_M}(Q(t)s+Q_l s)\right.\right.$$

$$\left.\left.+\frac{2Q_l Q(t)}{7c_M^2}\right]\right\}\delta s+\left[\frac{1}{10}\left(\frac{c_S}{T}+E_S\alpha_S^2\right)\theta_m^2+\frac{1}{21\lambda_S T}c_S^2\theta_m^2\ q\dot{q}\right]\delta q$$

$$=-\left[\frac{\rho l\theta_m}{T}+\frac{9E_M\alpha_M^2 Q_l^2}{10c_M^2 s^2}+\frac{1}{2}E_S\alpha_S^2\theta_m^2+\frac{\rho l}{\lambda_m}(2c_M\theta_m+\rho l)s\dot{s}\right]\delta s. \qquad (3.20)$$

§4. Heat Balance Integral.

We shall enlarge the heat balance integral introduced by Goodman.[7),8)] Refering to the energy equation:[10)]

$$-(h_i)_i=c\dot{\theta}+T\beta_{ij}\ \dot{e}_{ij}\ ,$$

h_i ······heat flow for x direction,

β_{ij}······numerical constant,

e_{ij}······strain tensor,

we define the enlarged heat balance integral

$$I=\int_0^s (c_M+TE_M\alpha_M^2)\theta_M\ dx+\int_s^{s+q}(c_S+TE_S\alpha_S^2)\theta_S\ dx\ . \qquad (4.1)$$

Differentiating on both sides of eq. (4.1) with respect to time provides:

$$\frac{dI}{dt}=(c_m+TE_m\alpha_m^2)\theta_m\ \dot{s}-(c_m+TE_m\alpha_m^2)\theta_m\ \dot{s}+\int_0^s(c_M+TE_M\alpha_M^2)\dot{\theta}_M\ dx$$

$$+\int_s^{s+q}(c_S+TE_S\alpha_S^2)\ \dot{\theta}_S\ \ dx$$

$$=\int_0^s\frac{\partial}{\partial x}\left(\lambda_M\frac{\partial\theta_M}{\partial x}\right)dx+\int_s^{s+q}\frac{\partial}{\partial x}\left(\lambda_S\frac{\partial\theta_S}{\partial x}\right)dx$$

$$=\lambda_m\left(\frac{\partial\theta_M}{\partial x}\right)_{x=s}-\lambda_m\left(\frac{\partial\theta_S}{\partial x}\right)_{x=s}-\lambda_M\left(\frac{\partial\theta_M}{\partial x}\right)_{x=0}$$

$$=Q(t)-\rho l\dot{s},$$

$$\therefore\qquad \frac{dI}{dt}=Q(t)-\rho l\dot{s}\ . \qquad (4.2)$$

Substituting eqs. (2.10) and (2.12) into eq. (4.1), we have

$$I=(c_M+TE_M\alpha_M^2)\left(\theta_m\ s+\frac{Q_l}{c_M}\right)+\frac{1}{3}\theta_m(c_S+TE_S\alpha_S^2)\ q\ . \qquad (4.3)$$

The equations (4.2) and (4.3) provide:

$$(c_M + TE_M \alpha_M^2)\left(\theta_m \dot{s} + \frac{Q(t)}{c_M}\right) + \frac{1}{3}\theta_m (c_S + TE_S \alpha_S^2)\dot{q} = Q(t) - \rho l\dot{s},$$

$$\therefore \left[(c_M + TE_M \alpha_M^2)\theta_m + \rho l\right]\dot{s} + \frac{1}{3}\theta_m (c_S + TE_S \alpha_S^2)\dot{q} = -\frac{TE_M \alpha_M^2}{c_M} Q(t). \qquad (4.4)$$

Integretion of eq. (4.4) provides

$$[(c_M + TE_M \alpha_M^2)\theta_m + \rho l]s + \frac{1}{3}\theta_m (c_S + TE_S \alpha_S^2)(q(t) - q(o))$$

$$= -\frac{TE_M \alpha_M^2}{c_M}\int_o^t Q(t)\ at. \qquad (4.5)$$

As in the previous paper,* the initial condition of $q(t)$ is given by

$$q(o) = \frac{\lambda}{Q_o}\theta_m \qquad (4.6)$$

The equation (4.5) provides the relation of δs and δq.

$$\delta q = -\frac{3}{\theta_m (c_S + TE_S \alpha_S^2)}\left[(c_M + TE_M \alpha_M^2)\theta_m + \rho l\right]\delta s. \qquad (4.7)$$

Inserting eq. (4.7) into eq. (3.20), we find

$$\frac{1}{2}\left(\frac{c_M}{T} + E_M \alpha_M^2\right)\left(\theta_m^2 s^2 - \frac{9Q_I^2}{5c_M^2}\right) + \frac{c_M^2}{2\lambda_M T}\left[2\theta_m^2 s\dot{s} - \frac{\theta_m}{5c_M}\left(Q(t)s + Q_I\dot{s}\right)\right.$$

$$\left. + \frac{2Q_I Q(t)}{7c_M^2}\right] s^2 + \left(\frac{\rho l\theta_m}{T} + \frac{1}{2}E_S \alpha_S^2 \theta_m^2\right) s^2 + \frac{9E_M \alpha_M^2 Q_I^2}{10c_M^2} + \frac{\rho l}{\lambda_m}(2c_M \theta_m + \rho l)s^3 \dot{s}$$

$$= \frac{3[(c_M + TE_M \alpha_M^2)\theta_m + \rho l]}{\theta_m (c_S + TE_S \alpha_S^2)}\left[\frac{1}{10}\left(\frac{c_S}{T} + E_S \alpha_S^2\right)\theta_m^2 + \frac{c_S^2 \theta_m^2}{21\lambda_S T} q\dot{q}\right] s^2. \qquad (4.8)$$

§5. Series Solution.

We shall derive the series solution of $s(t)$ from eqs. (4.4) and (4.8).

We set the assumption:

$$\frac{c_M}{T} \ll E_M \alpha_M^2 \quad \text{or} \quad \frac{Q}{c_M} \ll 1 \ (\text{near } t=0). \qquad (5.1)$$

Then eq. (4.8) becomes

$$\frac{1}{2}\left(\frac{c_M}{T} + E_M \alpha_M^2\right)\theta_m^2 + \frac{c_M^2}{2\lambda_M T}\left[2\theta_m^2 s\dot{s} - \frac{\theta_m}{5c_M}(Q(t)s + Q_I \dot{s}) + \frac{2Q_I Q(t)}{7c_M^2}\right]$$

$$+ \left(\frac{\rho l\theta_m}{T} + \frac{1}{2}E_S \alpha_S^2 \theta_m\right) + \frac{\rho l}{\lambda_m}(2c_M \theta_m + \rho l)s\dot{s}$$

* Y. Furuya: communication.

$$=\frac{3[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]}{\theta_m(c_S+TE_S\alpha^2_S)}\left[\frac{1}{10}\left(\frac{c_S}{T}+E_S\alpha^2_S\right)\theta^2_m+\frac{c^2_S\theta^2_m}{21\lambda_S T}q\dot{q}\right]. \tag{5.2}$$

We rewright eq. (4.4);

$$[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]\dot{s}+\frac{1}{3}\theta_m(c_S+TE_S\alpha^2_S)\dot{q}$$

$$=-\frac{TE_M\alpha^2_M}{c_M}Q(t). \tag{5.3}$$

From eq. (4.6) we have seen

$$q(o)=\frac{\lambda}{Q_o}\theta_m \tag{5.4}$$

Inserting $t=0$ into eq. (5.3) and (5.2), we have:

$$\dot{q}(o)=-\frac{3}{\theta_m(c_S+TE_S\alpha^2_S)}\left\{\frac{TE_M\alpha^2_M}{c_M}Q_o+[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]\dot{s}(o)\right\}, \tag{5.5}$$

$$\frac{1}{2}\left(\frac{c_M}{T}+E_M\alpha^2_M\right)\theta^2_m+\frac{c^2_M}{2\lambda_M T}\left(-\frac{\theta_m}{5c_M}\Delta\cdot s(o)+\frac{2\Delta\cdot Q_o}{7c^2_M}\right)+\left(\frac{\rho l\theta_m}{T}+\frac{1}{2}E_S\alpha^2_S\theta^2_m\right)$$

$$=\frac{3[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]}{\theta_m(c_S+TE_S\alpha^2_S)}\left[\frac{1}{10}\left(\frac{c_S}{T}+E_S\alpha^2_S\right)^{\bullet}\theta^2_m+\frac{c^2_S\theta^2_m}{21\lambda_S T}q(o)\dot{q}(o)\right]. \tag{5.6}$$

The equation (5.6) and (5.5) provide

$$\dot{s}(o)=\left\{\frac{c_M\theta_m\Delta}{10\lambda_M T}-\frac{3c^2[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]^2}{7\lambda_S T(c_S+TE_S\alpha^2_S)}\right\}^{-1}$$

$$\times\left\{\frac{1}{5}\left(\frac{c_M}{T}+E_M\alpha^2_M\right)\theta_m+\frac{\Delta\cdot Q_o}{7\lambda_M T}+\frac{7}{10}\rho l\theta_m+\frac{1}{2}E_S\alpha^2_S\theta^2_m\right.$$

$$\left.+\frac{3c^2_S E_M\alpha^2_M}{7\lambda_S c_M}\frac{[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]}{(c_S+TE_S\alpha^2_S)^2}Q_o\ q(o)\right\}. \tag{5.7}$$

Differentiating eq. (5.3) with respect to time and setting $t=0$, we have

$$\ddot{q}(o)=-\frac{3}{\theta_m(c_S+TE_S\alpha^2_S)}\left\{\frac{TE_M\alpha^2_M}{c_M}\dot{Q}(o)+[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]\ddot{s}(o)\right\}. \tag{5.8}$$

Also, differentiating eq. (5.3) with respect to time and setting $t=0$, we have

$$\frac{c^2_M}{2\lambda_M T}\left[2\theta^2_m s(o)^2-\frac{\theta_m}{5c_M}(2Q_o\dot{s}(o)+\Delta\cdot\ddot{s}(o))+\frac{2}{7c^2_M}(Q^2_o+\Delta\cdot\dot{Q}(o))\right]$$

$$+\frac{\rho l}{\lambda_m}(2c_M\theta_m+\rho l)s(o)^2=\frac{c^2_S\theta_m[(c_M+TE_M\alpha^2_M)\theta_m+\rho l]}{7\lambda_S T(c_S+TE_S\alpha^2_S)}(\dot{q}(o)^2+q(o)\ddot{q}(o)). \tag{5.9}$$

The equations (5.9) and (5.8) provide

$$\ddot{s}(o)=\left\{\frac{c_M\theta_m\Delta}{10\lambda_M T}-\frac{3c^2_S q(o)}{7\lambda_S T}\left[(c_M+TE_M\alpha^2_M)\theta_m+\rho l\right]\right\}^{-1}\times\left\{\frac{c^2_M\theta^2_m}{\lambda_M T}s(o)^2\right.$$

$$-\frac{c_M\ \theta_m Q_o}{5\lambda_M\ T}\ \dot{s}(o)+\frac{1}{7\lambda_M\ T}\ (Q_o^2+\Delta\cdot\dot{Q}(o))+\frac{\rho l}{\lambda_m}\ (2c_M\ \theta_m+\rho l)\dot{s}(o)^2$$

$$-\frac{c_S^2\ \theta_m\left[(c_M+TE_M\alpha_M^2)\theta_m+\rho l\right]}{7\lambda_S\ T(c_S+TE_S\alpha_S^2)}\ q(o)^2$$

$$+\frac{3c_S^2 E_M\alpha_M^2}{7\lambda_S\ T}\ \frac{\left[(c_M+TE_M\alpha_M^2)\theta_m+\rho l\right]}{(c_S+TE_S\alpha_S^2)^2}\ q(o)\ \dot{Q}(o)\Big\} \tag{5.10}$$

Continuing the same procrdure, we may reach the series solution:

$$s(t)=\dot{s}(o)t+\frac{1}{2!}\ \ddot{s}(o)t^2+\ \cdots\cdots\cdots. \tag{5.11}$$

$$q(t)=q(o)+\dot{q}(o)t+\frac{1}{2!}\ddot{q}(o)t^2\cdots\cdots\cdots. \tag{5.12}$$

§6. Discussion and Conclusion.

1. The quadratic formula of themal field is found; the temperature of the solid state is obtained by of the definition of thermal layer: [eqs. (2.10) and (2.12)].
2. The variational invariants are introduced: [eq. (3.1) and (3,2)]; the variational equation is formulated: [eq. (3.5)].
3. The enlarged heat balance integral is defined: [eq. (4.1)]. And differentiation of it with respect to time provides the equality: [eq. (4.2)].
4. The quadratic formula of temperature, variational equation and enlarged heat balance integral provide the system of differential equations of the position of melting line $s(t)$ and thickness of thermal layer $q(t)$: [eq. (4.4) and (4.8)].
5. We have made the assumption

$$\frac{c_M}{T} \ll E_M\alpha_M^2\ , \tag{6.1}$$

$$\frac{Q_I}{c_M} \ll 1 \quad near \quad t=0. \tag{6.2}$$

i) If the inequality (6.1) is given, we find an in equality

$$\theta\frac{c_M\delta\theta}{T} \ll E_M\,\alpha_M\,\theta\ \alpha_M\,\delta\theta\ .$$

The left hand side shows the thermal energy expended in the virtual displacement of the melting line. The right hand side shows the elastic energy expended in that virtual displacement.

ii) By the definition of Q_I, we find:

$$Q_I \ \rightarrow\ \Delta \qquad as \qquad t\ \rightarrow\ 0.$$

Therefore, eq. (6.2) is reduced to

$$\Delta\ \ll\ c_M$$

This shows material which has great heat capacity.

6. The system of equations (4.4) and (4.8), with the assumptipn (6.1) or (6.2) provide the series solution of s(t); [eq. (5.11)].

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